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Abstract—A general model of electrical oscillators under the influence of a periodic injection is presented. Stemming solely from the autonomy and periodic time variance inherent in all oscillators, the model’s underlying approach makes no assumptions about the topology of the oscillator or the shape of the injection waveform. A single first-order differential equation is shown to be capable of predicting a number of important properties, including the lock range, the relative phase of an injection-locked oscillator, and mode stability. The framework also reveals how the injection waveform can be designed to optimize the lock range. A diverse collection of simulations and measurements, performed on various types of oscillators, serve to verify the proposed theory.

Index Terms—Adler’s equation, impulse sensitivity function (ISF), injection locking, injection pulling, lock characteristic, lock range, oscillator, synchronization.

I. INTRODUCTION

By controlling the timing of events, oscillators generate the “heartbeat” of modern electronic systems. Their utility, however, is boosted significantly by their tendency to synchronize to external signals that are themselves periodic in time. Known as injection locking, this behavior has engendered numerous applications, including the recovery of timing information from data streams [2], clock distribution and jitter reduction for high-density input–output (I/O) links [3], [4], frequency division [5]–[7] and frequency multiplication [8], [9], the precise generation of quadrature or other multiphase signals [10]–[13], and the synchronization of elements in phased arrays [14], [15].

Despite its usefulness, the capability to lock becomes problematic when an oscillator is affected by unwanted perturbations in its environment. In particular, periodic disturbances that fail to lock the oscillator will instead corrupt the oscillator’s inherent periodicity or its “ability to tell time,” an undesirable phenomenon known as injection pulling.

In communication circuits, for example, the large-signal output of the power amplifier in a radio frequency transmitter can pull the oscillator generating the carrier, or pulling can occur between the receive and transmit local oscillators in a single-chip transceiver [16].

Despite the numerous studies that have been conducted on injection locking and pulling [16]–[41], models for describing this behavior have seen a disconnect between rigorous, mathematically-based approaches and design-oriented, physically-based analyses. While the analytical power of the former is often concealed by an inundation of abstract mathematical machineries, the accuracy and generality of the latter are constrained by the empirical nature of their ensuing derivations. This paper seeks to bridge that gap by developing a theory that:

1) leads to a conceptual understanding of the phenomenon of synchronization;
2) makes accurate, quantitative predictions about various aspects of the behavior of periodically disturbed oscillators;
3) yields practical insights into how to improve the implementation of systems that utilize injection locking.

The material is organized as follows. Section II introduces notation while Section III reviews existing models. Section IV introduces a simple thought experiment to motivate our approach before Section V formally develops a time-synchronous theory of injection locking and pulling. Finally, Section VI explores the design implications of the developed framework by analyzing the effect of the shape of the injection waveform on an oscillator’s locking properties. Other specialized topics, such as a large-injection model for LC oscillators, subharmonic/superharmonic injection locking, and the dynamics of injection pulling are treated in a companion paper [1].

II. PRELIMINARIES

A. Basic Setup and Notation

In general, the oscillation voltage of an oscillator can be written as

\[ v_{osc}(t) = [1 + A(t)] \cdot v_0[\omega_0 t + \phi(t)] \quad (1) \]

where \( \omega_0 \equiv 2\pi/T_0 \) is the (noiseless) free-running (angular) frequency of oscillation and \( v_0(\cdot) \) is a \( 2\pi \)-periodic oscillation waveform that captures the shape and size of the free-running oscillation voltage. Disturbances to the oscillator (internal or external, random or deterministic) can cause fluctuations in the waveform, \( A(t) \), and in the phase, \( \phi(t) \). As shown in Fig. 1(a), we are interested in the oscillator’s
behavior when the disturbance is a periodic injection of current
\[ i_{\text{inj}}(t) \equiv i_{\text{inj},0}(\omega_{\text{inj}} t) \quad (2) \]
where \( \omega_{\text{inj}} \equiv 2\pi/T_{\text{inj}} \) is the injection frequency and \( i_{\text{inj},0}(t) \) is a \( 2\pi \)-periodic injection waveform that captures the shape and size of the injection current.\(^1\)

From (1), the oscillator’s total phase \( \varphi(t) \) can be written as
\[ \varphi(t) \equiv \omega_{\text{inj}} t + \theta(t) \quad (3) \]
In injection locking and pulling scenarios, however, it is more useful to represent \( \varphi(t) \) in the frame of reference of the injection signal:
\[ \varphi(t) \equiv \omega_{\text{inj}} t + \Theta(t) \quad (4) \]
where \( \theta \) is the relative phase of the oscillator with respect to the injection. These two reference frames are depicted in Fig. 1(b). Notice that this new reference frame facilitates the treatment of the oscillator under a periodic injection by explicitly identifying the deviation from the injection, namely \( \Theta(t) \). Finally, the instantaneous frequency of oscillation, \( \omega_{\text{osc}} \), is defined as the time derivative of the total phase:
\[ \omega_{\text{osc}}(t) := \frac{d\varphi}{dt}. \quad (5) \]
Being an instantaneous quantity, this oscillation frequency is generally a function of time.

\( B. \) Definition of Injection Locking and Pulling

An injection-locked oscillator oscillates at the injection frequency.\(^2\) This requires the total phase, \( \varphi(t) \), to grow by exactly \( 2\pi \) every injection period. In light of (4), this means that the value of \( \theta \) cannot change by a net amount over the same time frame:
\[ \theta(t) = \theta(t + T_{\text{inj}}) \quad (6) \]
for all time \( t \). Alternatively, this can be represented as
\[ \frac{1}{T_{\text{inj}}} \int_{t}^{t+T_{\text{inj}}} \frac{d\theta}{dt} \, dt = 0. \quad (7) \]
\(^1\)It is also possible to consider current oscillation waveforms or voltage perturbations without a fundamental change to our development.
\(^2\)The more general cases of injection-locked frequency division and multiplication are addressed in a companion paper [1].

In other words, an injection-locked oscillator has an average oscillation frequency of \( \omega_{\text{inj}} \), although the instantaneous oscillation frequency \( \omega_{\text{osc}} \) may exhibit higher-order fluctuations within a single period.\(^3\) Fig. 1(c) shows a hypothetical example of the total phase of an injection-locked oscillator as a function of time.

Although the most general characterization of injection locking allows \( \theta \) to exhibit intra-period variations, a more limiting notion that is often used is
\[ \text{Injection Locked} \iff \frac{d\theta}{dt} = 0 \quad (8) \]
which corresponds to \( \theta \) being constant in time—a special case of (7). Justified rigorously through the technique of time-averaging [47], this narrower definition of injection locking simplifies the mathematical treatment significantly while providing many useful insights.

Under the framework set forth by (8), a question that arises is what this constant value of \( \theta \) is for an injection-locked oscillator. As we will see, the phase difference between the oscillator and the injection is not arbitrary—instead, \( \theta \) varies with the injection frequency in a specific manner. To quantify this important relationship, we define a function called the lock characteristic \( \Omega(\theta) \), which is equal to the frequency difference \( \Delta \omega := \omega_{\text{inj}} - \omega_{0} \) that results in an oscillation phase of \( \theta \) relative to the injection when locked:
\[ \Omega(\theta) := \Delta \omega \quad \text{for} \quad \frac{d\theta}{dt} = 0. \quad (9) \]
Note that the lock characteristic depends on both the oscillator and the injection waveform \( i_{\text{inj},0} \), as will become apparent in Section V.

As we can see, the lock characteristic represents the range of possible frequency differences that are achievable under lock. Therefore, we can use it to calculate the upper and lower lock ranges, which are the maximum and minimum values of \( \Delta \omega \) that the oscillator can lock to, respectively:
\[ \omega_{L}^+ = \max_{\theta} \Omega(\theta) \quad \text{and} \quad \omega_{L}^- = \min_{\theta} \Omega(\theta). \quad (10) \]
Finally, an oscillator that is not injection locked, either because the injection is outside of the lock range or because steady state has not yet been reached, is said to be injection pulled.
of the real and imaginary parts of $I$

Fig. 2. Schematic of an LC oscillator under sinusoidal injection used in deriving Adler’s equation.

### III. Survey of Existing Models

Perhaps the most well-known behavioral model for injection locking is Adler’s equation, developed in 1946 by the inventor Robert Adler [20], which describes the phase of an LC oscillator under the influence of a weak sinusoidal injection close to the free-running frequency. We present a simplified derivation here based on the schematic shown in Fig. 2.

In the free-running scenario, the oscillation voltage $v_{osc}(t)$ is sinusoidal at the LC tank’s resonant frequency

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad (11)$$

The loss of the LC tank, represented by the parallel resistance $R_P$, is compensated for by the nonlinear $-G_m$-transconductor, which generates a current whose fundamental component is in phase with $v_{osc}(t)$ and has an amplitude of $I_{osc}$. This leads to an oscillation amplitude of

$$V_{osc} = I_{osc} R_P. \quad (12)$$

Suppose a sinusoidal injection current $i_{inj}(t) = I_{inj} e^{j\omega_{inj}t}$ is injected into the oscillator.\(^4\) If we assume the injection is small in the sense that $I_{inj} \ll I_{osc}$, then its impact on the resistor current $i_R$ is negligible, leaving the oscillation amplitude unchanged:

$$v_{osc}(t) = V_{osc} e^{j(\omega_{inj}t + \theta)}. \quad (13)$$

Our objective here is to characterize the behavior of $\theta$, the oscillator’s relative phase. Writing KCL for the remaining currents, we get\(^5\)

$$i_{inj} = i_C + i_L \implies \frac{di_{inj}}{dt} = C \frac{d^2v_{osc}}{dt^2} + \frac{v_{osc}}{L}. \quad (14)$$

The intuition behind this approach is that the reactive current drawn by the LC tank when the oscillator operates away from resonance must be supplied by the injection current. Substituting for $i_{inj}$ and $v_{osc}$ and then multiplying through by $e^{-j(\omega_{inj}t + \theta)}$, we obtain

$$j\omega_{inj}I_{inj}e^{-j\theta} = \left[ C \left( j \frac{d^2\theta}{dt^2} - \left( \omega_{inj} + \frac{d\theta}{dt} \right)^2 \right) + \frac{1}{L} \right] I_{osc} R_P. \quad (15)$$

Further assuming that the injection frequency is near the free-running frequency ($|\Delta \omega_0| \ll \omega_0$) and that the oscillator’s phase varies slowly compared to the injection ($|d\theta/dt| \ll \omega_{inj}$), this equation’s real part can be simplified to Adler’s equation

$$\frac{d\theta}{dt} = \omega_0 - \omega_{inj} - \frac{\omega_0}{2Q} I_{inj} \sin \theta \quad (16)$$

where $Q$ is the tank’s quality factor:

$$Q = \frac{R_P}{\omega_0 L} = R_P \omega_0 C. \quad (16)$$

The steady-state solution of (15) yields a lock characteristic of

$$\Omega(\theta) = -\frac{\omega_0 I_{inj}}{2Q I_{osc}} \sin \theta \quad (17)$$

which results in a symmetric lock range $\omega_L = \omega_0^+ = -\omega_0^-$ of

$$\omega_L = \frac{\omega_0 I_{inj}}{2Q I_{osc}}. \quad (18)$$

Therefore, the lock range increases with the relative injection strength $I_{inj}/I_{osc}$ but varies inversely with the tank’s quality factor $Q$. Adler’s equation has since formed the basis for numerous other approaches to understanding injection locking and pulling in electrical oscillators [16], [22]–[29]. Adler’s equation suffers from a number of limitations, as it:

1) only holds for weak injection currents ($I_{inj} \ll I_{osc}$);
2) is only applicable to LC oscillators;
3) assumes the injection waveform is sinusoidal;
4) requires identification of the behavioral parameters $Q$ and $I_{osc}$, which may not be easy to accurately determine in modern integrated oscillators due to parasitics from the active devices and the layout;
5) predicts a symmetrical lock range $\omega_L^+ = -\omega_L^-$, which is generally not true [4], [28].

Forgoing the restriction of the analysis to weak injections will be examined in substantially more detail in a companion paper [1]. To address the second limitation, a variety of behavioral approaches for analyzing injection locking in ring oscillators have been developed.\(^6\) One methodology that has gained considerable traction treats each stage of the ring as a RC-delay cell driven by an idealized, nonlinear transconductor [28], [32], [39], [40]. This physically-based model is useful to designers as it permits analysis even when multiple nodes of the ring oscillator are injected into, a technique which can widen the lock range significantly if done at the proper relative phases [1], [28], [39], [40]. Another interesting model, based on the so-called phase domain response (PDR), is introduced in [41]; although not confined to a particular oscillator topology, the lack of time-synchronicity in its underlying mathematical framework unfortunately leads to flawed predictions in many situations [48, Appendix B].

Finally, we also wanted to mention the work of Maffezzoni [34], as it comes the closest to our model. Specifically, their analysis uses the impulse sensitivity function (ISF) (from the periodically time-varying oscillator phase noise model of [42]–[45]) to derive the lock range of an arbitrary oscillator. No assumptions are made about the shape of the injection or the relative harmonic between the injection and the oscillation (i.e., the injection-locked oscillation frequency generally satisfies $N\omega_{osc} = M\omega_{inj}$, where $M$ and $N$ are positive integers). However, their work exposes only the tip of the iceberg in terms of the wide array of properties and behaviors that can be predicted using the ISF.\(^6\)

\(^4\)We use phasors to simplify the subsequent algebra. It is understood that the actual, physical current injection can be an arbitrary linear combination of the real and imaginary parts of $I_{inj} e^{j\omega_{inj}t}$.

\(^5\)Due to Adler’s weak-injection assumption, the resistor and transconductor currents roughly cancel and therefore do not appear in (14). See [16], [20], [27], [28] for more details.

\(^6\)Relaxation oscillators have received much less attention in this regard, as ring oscillators are far more commonly used in integrated settings due to their ease of implementation.
IV. Locking to an Impulse Train

The behavior of an ideal LC oscillator driven by an impulse train provides valuable insights into certain properties of injection locking. Consider the setup of Fig. 3(a), which shows an ideal parallel LC oscillator in parallel with an impulse train injection current \( i_{\text{inj}}(t) \), which periodically dumps a fixed amount of charge \( q_{\text{inj}} \) onto the capacitor \( C \) at a period of \( T_{\text{inj}} = 2\pi/\omega_{\text{inj}} \). Note that the pertinent part of the oscillatory behavior comes from the LC tank; the parallel resistance and the transconductor serve only to set the oscillation amplitude. It is possible for each injection to induce a shift in only the oscillation period without affecting the frequency or the amplitude of the oscillation. However, the oscillation period \( T_{\text{inj}} \) and the frequency \( \omega_{\text{inj}} \) are not to scale, as they have been enlarged for illustrative purposes.

Next, one can analyze the balance between the LC oscillator’s tank loss and its energy restoration mechanism to obtain the identity shown in Fig. 3(a):

\[
\omega_0 q_{\text{max}} = Q I_{\text{osc}}
\]  

where the tank’s quality factor \( Q \) is given by (16). Also, for the sake of argument, let us look at the amplitude of the fundamental component of the impulse train:

\[
I_{\text{inj}} = \frac{2q_{\text{inj}}}{T_{\text{inj}}}
\]  

Now, substituting (22) and (23) into (21) curiously yields an intuitive basis with which to understand the time-synchronous model presented next.

V. Time-Synchronous Model

A. Review of the Impulse Sensitivity Function (ISF)

The response of an oscillator’s excess phase \( \phi(t) \) to an arbitrary injection of current \( i_{\text{inj}}(t) \) can be calculated using a superposition integral

\[
\phi(t) = \int_{-\infty}^{\infty} h_\phi(t, \tau)i_{\text{inj}}(\tau) \, d\tau
\]  

where \( h_\phi(t, \tau) \) is the excess phase at time \( t \) due to injecting a unit impulse of current at time \( \tau \). As shown in Fig. 4, this time-varying impulse response \( h_\phi(t, \tau) \) can be written as

\[
h_\phi(t, \tau) = \tilde{\Gamma}[\phi(\tau)]u(t - \tau).
\]  

The unit-step function captures the oscillator’s autonomy or lack of an absolute time reference—any perturbation in its phase appears immediately and will persist indefinitely. On the other hand, the periodically time-varying nature of the oscillator is captured by \( \tilde{\Gamma}(\phi) \), which is equal to the phase shift induced by injecting a discrete amount of charge as a function of the oscillator’s total phase \( \phi \) [see (3) and Fig. 1] at the time

\[\Delta \phi = \pm \frac{q_{\text{inj}}}{q_{\text{max}}}. \]

This phase shift is related to the period change \( \Delta T \) and the frequency shift \( \Delta \omega \) via

\[
\frac{\Delta \phi}{2\pi} = \frac{\Delta T - \Delta \omega}{\omega_0} = \frac{\Delta \omega}{\omega_0}
\]

which results in

\[\Delta \omega = \frac{\Delta \phi}{T_{\text{inj}}} = \pm \frac{1}{T_{\text{inj}}} \frac{q_{\text{inj}}}{q_{\text{max}}}. \]

In general, when \( q_{\text{inj}} \) is comparable to \( q_{\text{max}} \), the phase shift is given by

\[
\Delta \phi = \pm 2 \sin^{-1} \left[ \frac{q_{\text{inj}}}{2q_{\text{max}}} \right].
\]

Thus, the maximum possible decrease or increase in the period that can be induced without changing the amplitude is achieved with \( q_{\text{inj}} = 2q_{\text{max}} \) and is equal to \( \Delta T = \pi T_0/2 \). This leads to \( \Delta \omega = \omega_0 \) and \( \Delta \omega = -\omega_0/3 \), respectively. It is noteworthy that this frequency shift, and therefore the “lock range” even for the case of an ideal LC oscillator in this thought experiment, is asymmetrical.

The logic behind this step will become clearer later (and is also discussed extensively in [1]). For now, we will rely on the intuition that the narrowband nature of the LC tank filters out all the components of the injection not in the vicinity of resonance.

9The extension of this thought experiment to injections which no longer preserve the amplitude is dealt with in a companion paper [1, Sec. III-B].

The underlying assumption of linearity here will be discussed shortly.
of injection,\(^{12}\) normalized by the amount of injected charge. Known as the impulse sensitivity function (ISF), \(^{\Gamma}(\cdot)\) has units of \([1/\text{Coulomb}]\) and is periodic with a period of \(2\pi\). Methods for calculating the ISF can be found in [45] and [46].

Note that the unit-less ISF defined in [42]–[45], which is normalized by an additional factor—the inverse of the maximum charge swing \(q_{\text{max}}\) across the injection terminals:

\[
\tilde{\Gamma}(x) \equiv \frac{\Gamma(x)}{q_{\text{max}}}. \tag{26}
\]

However, our version of the ISF, namely \(\hat{\Gamma}(\cdot)\), will be more convenient for us to work with.

### B. A Differential Equation for the Oscillator’s Phase

Substituting (25) into (24), we get

\[
\phi(t) = \int_{-\infty}^{t} \hat{\Gamma}(\varphi(\tau)) i_{\text{inj}}(\tau) \: d\tau. \tag{27}
\]

Differentiating with respect to time,

\[
\frac{d\phi}{dt} = \hat{\Gamma}(\varphi(t)) i_{\text{inj}}(t). \tag{28}
\]

Transforming our frame of reference to that of the injection [see (3) and (4) as well as Fig. 1(b)] allows us to rewrite this differential equation in terms of \(\theta\):

\[
\frac{d\theta}{dt} = \omega_0 - \omega_{\text{inj}} + \hat{\Gamma}(\omega_{\text{inj}} t + \theta) i_{\text{inj}}(t). \tag{29}
\]

The term \(\omega_0 - \omega_{\text{inj}}\) reflects the fact that in the absence of the injection’s influence, the phase difference \(\theta\) between the oscillator and the injection naturally grows at a constant rate given by the frequency difference between them. Next, we appeal to the theory of time-averaging [47] to obtain\(^{13}\)

\[
\frac{d\theta}{dt} = \omega_0 - \omega_{\text{inj}} + \frac{1}{T_{\text{inj}}} \int_{T_{\text{inj}}} \hat{\Gamma}(\omega_{\text{inj}} t + \theta) i_{\text{inj}}(t) \: dt. \tag{30}
\]

Note that \(\theta\) is to be treated as a constant within the averaging integral on the right-hand side. This is the basic differential equation that governs the oscillator’s phase \(\theta\) in the presence of a periodic external injection of current \(i_{\text{inj}}(t)\). We will refer to this important equation as the pulling equation.

Fig. 5 illustrates the process described by the pulling equation. (The notation in this diagram relies on the fact that averaging is independent of time scale, and so averaging \(\hat{\Gamma}(\omega_{\text{inj}} t + \theta) i_{\text{inj}}(t)\) over \(T_{\text{inj}}\) is the same as averaging \(\hat{\Gamma}(x + \theta) i_{\text{inj}}(0)\) over \(2\pi.\)) If this process reaches static equilibrium, the oscillator becomes injection locked.

The periodicity of the ISF and the injection allows us to expand them using Fourier series:

\[
\hat{\Gamma}(x) = \frac{\hat{\Gamma}_0}{2} + \sum_{n=1}^{\infty} \left|\hat{\Gamma}_n\right| \cos(nx + \omega_n t) \tag{31a}
\]

\[
i_{\text{inj}}(t) = \frac{i_{\text{inj}}(0)}{2} + \sum_{n=1}^{\infty} \left|\hat{i}_{\text{inj},n}\right| \cos(n\omega_{\text{inj}} t + \Omega_{\text{inj},n}) \tag{31b}
\]

This enables us to evaluate the averaging integral in (30) in terms of these Fourier series coefficients:

\[
\frac{d\theta}{dt} = \omega_0 - \omega_{\text{inj}} + \frac{1}{2} \left[ \frac{i_{\text{inj}}(0)\hat{\Gamma}_0}{2} + \sum_{n=1}^{\infty} \left|\hat{i}_{\text{inj},n}\right| \cos(n\theta + \omega_n t - \Omega_{\text{inj},n}) \right]. \tag{32}
\]

As we will see, this representation is easier to handle in many scenarios. Moreover, it shows how the various spectral components of the injection current are filtered by those of the ISF, a concept emphasized in Fig. 6. Put another way, the coefficients \(\hat{\Gamma}_n\) determine the impact that the various harmonics of the injection signal have on the phase behavior and locking properties of the oscillator. For an injection that contains most of its power at the \(N\)th harmonic, for example, the \(N\)th path of this block diagram will dominate the pulling process.\(^{14}\)

\(^{12}\)Several examples of this time variance can be found in [42], [45], [48].

\(^{13}\)Although we could also have applied the general definition of injection locking (7) to the un-averaged equation (29) directly, the analysis of differential equations which have an explicit dependence on time is considerably more difficult and, in this case, obscures the insight into the lock characteristic that our approach exposes.

\(^{14}\)This discussion hints at superharmonic locking, or injection-locked frequency division, which will be formally addressed in a companion paper [1].
C. The Lock Characteristic and the Lock Range

Recall from (9) that the lock characteristic $\Omega(\theta)$ is defined as the relationship between $\Delta \omega$ and $\theta$ when the oscillator is locked. Therefore, we can see from our pulling equation (30) that

$$\Omega(\theta) = \frac{1}{T_{\text{inj}}} \int_{T_{\text{inj}}} \tilde{f}(\omega_{\text{inj}} t + \theta) i_{\text{inj}}(t) \, dt. \quad (33)$$

Although the extreme values of the lock characteristic correspond to the lock range, we will soon see that the lock characteristic captures practically all of the essential information about an oscillator’s injection locking and pulling behavior.

The concept of the lock characteristic is perhaps best understood in the time domain. By superposing the ISF and the injection current on a single plot, Fig. 7 illustrates how the relative phase $\theta$ controls the frequency shift for a hypothetical oscillator. The accumulation of area under the product of these two waveforms represents the total phase shift induced by the injection. Although this accumulation does not occur at a constant rate, the area per period is a fixed quantity, which gives rise to the frequency shift captured by the technique of time-averaging. This is expressed mathematically by (33). Three specific cases are shown here. The upper and lower edges of the lock range, which are depicted in Fig. 7(a) and (b), correspond to the values of $\theta$ that result in the largest positive area and largest negative area per period, respectively. On the other hand, the injection in Fig. 7(c) occurs at the free-running frequency; consequently, the phase difference $\theta$ is such that the positive and negative areas exactly cancel over a single period, yielding no net phase shift per period and no frequency shift on average. One might have noticed that these examples also expose interesting implications concerning the design of the shape of the injection waveform, which we will discuss in detail in Section VI.

The depiction of $\theta$ as being proportional to the difference between the zero-crossing times is just for illustrative purposes. Based on our setup, $\theta$ should actually be calculated as the phase difference between the fundamental components of the oscillation voltage and the injection current.

In effect, the oscillator’s injection-to-phase relationship acts as a periodically time-varying integrator. This behavior enables a periodic injection to contribute a fixed phase shift per period, which can result in the frequency shift necessary to synchronize the oscillator to the injection.

D. Sinusoidal Injection

Of particular importance is the special case of a sinusoidal injection current $i_{\text{inj}}(t) = I_{\text{inj}} \cos(\omega_{\text{inj}} t)$. This results in a lock characteristic of

$$\Omega(\theta) = \frac{1}{2} |i_{\text{inj}}| \tilde{f}_1 \cos(\theta + \tilde{f}_1) \quad (34)$$

which yields a lock range of [34]

$$\omega_L = \frac{1}{2} |i_{\text{inj}}| \tilde{f}_1. \quad (35)$$

E. Discussion of Linearity

The system at the core of our model—whose input is the injection current $i_{\text{inj}}(t)$ and output is the excess phase $\phi(t)$—is in general nonlinear; most systems in nature are. However, when an oscillator is injected with a discrete amount of charge $q_{\text{inj}}$ that is much smaller in magnitude than the maximum charge swing $q_{\text{max}}$ across the capacitance between the injection terminals, it has been observed that the incurred phase shift scales proportionally with $q_{\text{inj}}$[42]. This allows us to assess the linearity of the oscillator’s current-to-phase relationship, which is what we truly care about. We surmise that as long as the total amount of excess charge accumulated at the injection node is also small compared to $q_{\text{max}}$, linearity prevails and (24) is valid.\(^{17}\)

15The depiction of $\theta$ as being proportional to the difference between the zero-crossing times is just for illustrative purposes. Based on our setup, $\theta$ should actually be calculated as the phase difference between the fundamental components of the oscillation voltage and the injection current.

16For a non-sinusoidal waveform, the swing $q_{\text{max}}$ is generally defined as half of the peak-to-peak amplitude.

17From another perspective, note that the ISF is actually the gradient of the oscillator’s phase along its limit cycle with respect to injected charge [35], [42], and so an ISF-based model represents a linearization of the oscillator’s behavior about its free-running operation or its limit cycle in state space. A more extensive discussion of linearity in this context, as well as an example of when it does not hold, can be found in [48].
For ring and relaxation oscillators, strong amplitude-limiting mechanisms cause any excess charge to decay within a tiny fraction of the oscillation period. Taking this into account and assuming a sinusoidal injection of amplitude \( I_{\text{inj}} \), for example, one can show that linearity requires

\[
I_{\text{inj}} \ll I_{\text{max}}
\]

where

\[
I_{\text{max}} := \omega_0 q_{\text{max}}.
\]

Note that the effect that this linearization has on our model lies only in the linearity of the lock characteristic \( \Omega(\theta) \) with respect to the injection current \( i_{\text{inj}}(t) \); we are not assuming that oscillators are linear systems in general, nor are we neglecting the nonlinear behavior of any active devices. In fact, the various nonlinearities within the oscillator play a significant role in our model through their impact on the ISF. In this paper, we will demonstrate through simulations and measurements that the linearity upon which our model is predicated holds for a wide range of practical injection strengths.

### F. Stability

The periodicity of the lock characteristic \( \Omega(\theta) \) implies the existence of multiple steady-state solutions for \( \theta \) given an injection frequency within the lock range. However, not all of these solutions, or modes of locking, are stable. To see this, first use the lock characteristic of (33) to rewrite the pulling equation in the following way:

\[
\frac{d\theta}{dt} = -\Delta\omega + \Omega(\theta).
\]

Now let \( \theta(t) = \theta_0 + \hat{\theta}(t) \), where \( \theta_0 \) is the injection-locked phase and \( \hat{\theta} \ll 1 \) is some small disturbance. Then, utilizing a first-order Taylor series approximation yields the following differential equation for \( \hat{\theta}(t) \):

\[
\frac{d\hat{\theta}}{dt} = \Omega'(\theta_0) \cdot \hat{\theta}.
\]

For \( \theta_0 \) to be a stable relative phase under lock, \( \hat{\theta} \) must decay to 0 as time progresses. That is,

\[
\text{\( \theta_0 \) is a stable mode \iff \Omega'(\theta_0) = \frac{\partial \Delta\omega}{\partial \theta} \bigg|_{\theta=\theta_0} < 0.\)
\]

In other words, \( \theta_0 \) is stable if and only if the lock characteristic has a negative slope at \( \theta_0 \).

The solution to (39) follows an exponential decay dynamic: \( \hat{\theta}(t) \propto e^{-\tau_p} \), where we define a pull-in time \( \tau_p \) and a pull-in frequency \( \omega_p \) by

\[
\frac{1}{\tau_p} \equiv \omega_p := -\Omega'(\theta_0).
\]

In other words, the pull-in time is equal to the negative of the inverse of the slope of the lock characteristic. By differentiating the lock characteristic expressed in terms of the ISF’s Fourier series coefficients, we can also arrive at the decomposition for calculating \( \omega_p \) shown in Fig. 8.

It is instructive to think about the dynamics of injection locking and pulling from another perspective. Using (4) and (5), one can rewrite (38) as

\[
\Omega(\theta) = \omega_{\text{osc}} - \omega_0.
\]

The utility of this representation lies in its extension of the concept of the lock characteristic to the dynamics of injection pulling. The key insight here is that the lock characteristic controls the relationship between \( \theta \) and the shift in the oscillation frequency, \( \omega_{\text{osc}} - \omega_0 \), even when the oscillator is not locked. The resultant time-varying \( \theta \) then causes the lock characteristic to produce a “new” instantaneous frequency shift. This feedback-based perspective is emphasized by redrawing the block diagram of Fig. 5 in the form of Fig. 9. The averaging of the product \( i_{\text{inj}}(x - \theta) \cdot \hat{\Gamma}(x) \) is abstracted away into a single feedback block: the lock characteristic \( \Omega(\theta) \), which is nonlinear with respect to \( \theta \). The frequency difference \( \Delta\omega \) is portrayed as an input that the system is trying to match.

More insight can be obtained, particularly into the notion of stability, from the graph of the lock characteristic \( \Omega(\theta) \). A hypothetical example is provided in Fig. 10. Based on (38), we know that \( \theta \) will increase (i.e., move to the right) when \( \Omega(\theta) \) is greater than (i.e., above) the frequency difference \( \Delta\omega \) and vice versa. This dynamic gives rise to the directions indicated by the arrows drawn on the lock characteristic near where it crosses \( \Delta\omega \). Therefore, although two distinct steady-state solutions to \( \theta \) exist for the frequency difference \( \Delta\omega \), only \( \theta_0 \) corresponds to a stable mode.

### G. Injection Pulling

On the other hand, if the frequency difference is outside of the lock range (\( \Delta\omega_{\text{pull}} \) in Fig. 10), then \( \theta \) will continue to change indefinitely and the oscillator will never lock. The ensuing repeated traversal by \( \theta \) along the periodic lock characteristic gives rise to the “beats” observed in an injection-pulled oscillator. This is illustrated further in Fig. 11, which shows the interplay between \( \Omega(\theta) \) and \( \theta \) over time for an oscillator pulled by an injection above the lock range. By setting the instantaneous frequency shift \( \omega_{\text{osc}} - \omega_0 \) that the oscillator
experiences, the lock characteristic $\Omega(\theta)$ dictates how fast $\theta$ changes, as we can see from (38).

Observe how the oscillator spends a considerable amount of time “trying” to lock, during which $\theta$ remains relatively constant. This happens when $\Omega(\theta)$ is effectively situated at the upper edge of the lock range—close to the injection’s frequency difference $\Delta \omega$ as it can get—and so $|d\theta/dt|$ is

at its minimum. Unable to actually reach $\Delta \omega$, however, the oscillator never quite “catches up” to the injection and eventually slips behind by an entire cycle. The amount of time this whole process takes to repeat itself, wherein $\theta$ varies by $2\pi$, is known as the beat period $2\pi/\omega_b$.

Closed-form expressions for $\theta(t)$ (and $\omega_o$) are given in a companion paper [1] for the case of a sinusoidal injection.

H. Simulation Results

Here, we compare the simulated lock characteristics due to sinusoidal injection currents$^{19}$ of various amplitudes against the theoretical lock characteristics computed from (34). Two different oscillators are considered, both free-running at roughly $f_0 = 1$ GHz: the 6-stage differential ring oscillator shown in Fig. 12, and the Bose relaxation oscillator$^{20}$ shown in Fig. 13.$^{21}$ The ISFs were calculated through direct simulation of the impulse response [45].

By plotting the lock characteristic, we can see how the injection frequency $f_{inj}$ varies with the locked oscillator’s relative phase $\theta$ over the entire lock range. The vertical minimum and maximum of this plot therefore correspond to the lower and upper lock ranges, respectively. Unstable parts of the theoretical lock characteristic are distinguished by portraying them with dashed lines. The simulated lock characteristic was obtained through repeated transient simulations: first the lock range was determined by inspection (to a 1-MHz accuracy), and then the injection frequency $f_{inj}$ was swept over the lock range and $\theta$ was computed for each sweep point. It is noteworthy that even though deviations between the simulated and theoretically predicted lock characteristics become more pronounced for stronger injection currents, they still track reasonably closely for injection strengths $f_{inj}$ comparable to $f_{max}$. Note that only a very limited set of illustrative examples is provided here—a more extensive collection of lock characteristic comparison plots for other types of oscillators can be found in [48, Sec 4.6].

$^{18}$In light of this repeated behavior, one might suspect that the oscillation voltage of an injection-pulled oscillator is periodic with the beat frequency $\omega_b$. Unfortunately, this is not correct unless the injection frequency $\omega_{inj}$ is a multiple of $\omega_b$, which is not true in general. Therefore, an oscillator’s periodicity is fundamentally corrupted by injection pulling.

$^{19}$Non-sinusoidal injections will be explored in Section VI.

$^{20}$Because of the Bose oscillator’s simple RC-based charge–discharge behavior, closed-form expressions can be obtained for its oscillation waveform and ISF. Assuming $V_{max}^+ = -V_{max} = V_{max}$ and $V_{max}^- = V_{max}/2$; for example, we have

$$v_0(\phi) = \frac{V_{max}}{2} \left\{ \begin{array}{ll}
2 - 3(\phi - \pi)/\pi, & 0 < \phi \leq \pi \\
2 - 3(\phi - 2\pi)/\pi, & \pi < \phi < 2\pi
\end{array} \right. \quad \text{(41)}$$

and

$$\tilde{I}(\phi) = \frac{1}{q_{max}} \left\{ \begin{array}{ll}
\frac{\pi}{\ln 3}, & 0 < \phi < \pi \\
\frac{\pi}{\ln 3}, & \pi < \phi < 2\pi
\end{array} \right. \quad \text{(41)}$$

where $q_{max} = C V_{max}/2$. One can also show via a state-space analysis that the ISF and the oscillation waveform are related through

$$\tilde{I}(\phi) = \frac{1}{C \cdot v_0(\phi)} \quad \text{(42)}$$

$^{21}$LC oscillators under injection possess certain interesting properties and are therefore considered in significantly more detail in a companion paper [1].
I. Experimental Results

Finally, we present lock range measurements on a variety of integrated oscillators fabricated in a 65-nm bulk CMOS process. The measurements are compared against the theoretical predictions made by our model, where the ISFs were obtained from simulation of the extracted oscillators after layout.\textsuperscript{22} For each oscillator, we measured the lock range at

\textsuperscript{22}Plots of these ISFs can be found in [48, Sec 4.7].
Fig. 14. Measured lock ranges of several ring oscillators. The 3-stage and 17-stage rings are single-ended inverter-chain ring oscillators, where a capacitor loads the output of each inverter and the injection is applied at one of the outputs. The measured free-running oscillation frequencies are 1.32 GHz for the 6-stage differential ring and 1.09 GHz for both the 3-stage and 17-stage single-ended rings.

Fig. 15. Measured lock ranges of a couple relaxation oscillators. The schematic of the differential NMOS astable multivibrator is shown for reference. The measured free-running oscillation frequencies are 11.9 MHz for the Bose oscillator and 874 MHz for the astable multivibrator.

Fig. 16. Die photograph of the chip, with dimensions 1 × 1 mm².

various sinusoidal injection amplitudes. The predicted lock range is therefore given by (35).

Measurement results for a short, a medium, and a long ring oscillator are presented in Fig. 14, whereas measurement results for two different relaxation oscillators are shown in Fig. 15. Observe how the accuracy of the theoretically predicted lock range (and therefore the assumption of linearity) still prevails even for a broad range of practical injection strengths, even those that are comparable to $I_{max}$. Furthermore, it is noteworthy that the way in which the measured and predicted lock ranges deviate for larger injection amplitudes is reproducible in simulation as well [48]. A die micrograph of the fabricated oscillators is shown in Fig. 16.

23To account for possible measurement error, each oscillator was measured three separate times. Error bars depicting the entire range of measurements for each data point are shown in black. (The error bars for most data points are not noticeable.)
VI. DESIGN IMPLICATIONS—SHAPING THE INJECTION

As alluded to in Fig. 7, the shape of the injection seems to play a role in the frequency shift that is affected. In this section, we will explore how this phenomenon can be used to optimize the lock range. However, since the lock range increases with the injection strength, we must first constrain the size of the injection current in a meaningful way. Although this may be straightforward to do for relatively simple waveforms like sine or square waves, it becomes difficult to ascertain the injection “amplitude” when a complicated assortment of harmonics is present. A more universal measure of the injection size which accounts for power in all harmonics is the root-mean-square (rms) of the injection current

\[
I_{\text{rms}} \equiv \sqrt{\langle i_{\text{inj}}^2 \rangle} = \sqrt{\frac{1}{T_{\text{inj}}} \int_{T_{\text{inj}}} i_{\text{inj}}^2(t) \, dt}. \tag{43}
\]

At first glance, the rms injection current might also seem to represent the average power injected into the oscillator, but this assumes a fixed load, which is rarely the case in practice for the input impedances of actual oscillators. However, from a different and more practical perspective, \(I_{\text{rms}}\) actually serves as a good measure of the average power consumption of the injection circuitry itself.

To understand why, consider the differential transistor pair in Fig. 17, which commutates a static tail bias current \(I_{\text{bias}}\).

In the most efficient scenario, the differential injected current \(i_{\text{inj}}\) strictly alternates between \(\pm I_{\text{bias}}\). This injection current has an rms amplitude of \(I_{\text{rms}} = I_{\text{bias}}\), which is proportional to the static power consumption of the injection circuit: \(I_{\text{bias}}V_{\text{DD}}\). In reality, however, the circuit cannot transition between \(\pm I_{\text{bias}}\) instantaneously, resulting in time periods where the circuit is injecting less current. Thus, the average power consumption of the injection circuitry is usually at least \(I_{\text{rms}}V_{\text{DD}}\).

In summary, the rms injection current is a meaningful metric to consider because of its physical significance from a design standpoint—the minimum average power drawn by the injection circuitry scales with \(I_{\text{rms}}\)—and because it serves as an unambiguous definition of the injection amplitude regardless of the shape of the injection waveform. With this in mind, we are now in a position to think about how we can broaden the lock range by shaping the injection current for a fixed “injection power,” or more precisely, a fixed rms amplitude \(I_{\text{rms}}\).

One can show using the Cauchy–Schwarz inequality that the maximum lock range is obtained when the injection waveform is proportional to the ISF \([49]\). Thus, the optimal injection waveform \(i_{\text{inj},0}\) is given by

\[
i_{\text{inj},0}(x) = \pm \frac{I_{\text{rms}}}{\Gamma_{\text{rms}}} \tilde{\Gamma}(x) \tag{44}
\]
where the positive solution optimizes the upper lock range and the negative solution optimizes the lower lock range. The optimal (absolute) lock range is therefore easily calculated to be

$$\omega^* L = I_{\text{rms}} f_{\text{rms}}.$$  \hspace{1cm} (45)

Effectively, the lock range is maximized when the injection has the same shape as the ISF. This idea is illustrated conceptually in the cartoon of Fig. 18. After all, the ISF is a measure of the sensitivity of the oscillator’s phase to external disturbances as a function of when the disturbance is applied. Hence, an injection current which “looks like” the ISF is more active at points along the oscillation cycle that are more sensitive to the injection. We can also extend this intuition to a more physical level. For a given injection node, the oscillator’s phase is more impressionable—and the ISF is larger—when the voltage at that node is changing more rapidly.\(^{24}\) It therefore makes sense that displacing charge at the injection node during those times will advance or retard the oscillation more effectively.

Fig. 19 demonstrates this principle in action for a 17-stage single-ended ring oscillator. As we can see, the ISF consists of two tall, narrow pulses, which correspond to the sharp upward and downward transitions in the ring’s node voltages. By emulating this shape in the injection waveform, the lock range is almost doubled compared to a sinusoidal injection of the same power. In particular, Fig. 19(b) shows how the pulses in the injection target the transitions in the oscillation voltage at the edges of the lock range. At the lower edge, for example, notice how the injection current makes the voltage transitions less steep, thereby slowing down the oscillation.

In closing, we point out that there undoubtedly are scenarios where it is instead desirable to minimize the lock range in order to reduce coupling effects between oscillators, for example. This problem, which may warrant further investigation, can also be attacked with the framework developed in this paper.

VII. CONCLUSION

This paper presented a time-synchronous theory of injection locking and pulling in electrical oscillators, applicable to oscillators of any topology and periodic injections of arbitrary shape. A general mathematical characterization of autonomy and periodic time variance—the modus operandi of any oscillator—was used to derive a first-order differential equation for the time-domain behavior of the phase of a periodically disturbed oscillator. The framework revealed that the lock range is enhanced by matching the shape of the injection waveform to that of the oscillator’s ISF. Various simulation and measurement results support the proposed theory.

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